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Consistent $SO(6)$ Reduction Of Type IIB Supergravity on S^5

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ABSTRACT

Type IIB supergravity can be consistently truncated to the metric and the self-dual 5-form. We obtain the complete non-linear Kaluza-Klein S^5 reduction Ansatz for this theory, giving rise to gravity coupled to the fifteen Yang-Mills gauge fields of $SO(6)$ and the twenty scalars of the coset $SL(6,\mathbb{R})/SO(6)$. This provides a consistent embedding of this subsector of $N = 8$, $D = 5$ gauged supergravity in type IIB in $D = 10$. We demonstrate that the self-duality of the 5-form plays a crucial role in the consistency of the reduction. We also discuss certain necessary conditions for a theory of gravity and an antisymmetric tensor in an arbitrary dimension D to admit a consistent sphere reduction, keeping all the massless fields. We find that it is only possible for $D = 11$, with a 4-form field, and $D = 10$, with a 5-form. Furthermore, in $D = 11$ the full bosonic structure of eleven-dimensional supergravity is required, while in $D = 10$ the 5-form must be self-dual. It is remarkable that just from the consistency requirement alone one would discover $D = 11$ and type IIB supergravities, and that $D = 11$ is an upper bound on the dimension.

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1 Introduction

Non-trivial Kaluza-Klein sphere reductions of supergravity theories have been studied in a number of contexts. Long ago it was demonstrated [1] that the linearised analysis of the zero-mode fluctuations of the S^7 reduction of $D = 11$ supergravity [2, 3] can be extended to a fully non-linear and consistent embedding of maximal $D = 4$ gauged $SO(8)$ supergravity in $D = 11$. The fact that the truncation to the zero-mode sector is consistent, despite the non-linearities of the theory, is rather non-trivial since there appears to be no group-theoretic understanding of why it should work.

More recently, a similar, although more explicit, demonstration of the consistency of the S^4 reduction from $D = 11$ to the maximal $D = 7$ gauged $SO(5)$ supergravity was given [4, 5]. The consistent reduction of massive type IIA supergravity on a locally S^4 space, to give the $N = 2$ gauged $SU(2)$ supergravity in $D = 6$, has also been obtained [6]. In addition, certain reductions to truncations of the maximal gauged supergravities in various dimensions have also been constructed. These have the advantage of being considerably simpler than the maximal theories, allowing the reduction Ansatz to be presented in a more explicit form. Cases that have been constructed include the $N = 2$ gauged $SU(2)$ supergravity in $D = 7$ [7]; the $N = 4$ gauged $SU(2) \times U(1)$ supergravity in $D = 5$ [8]; the $N = 4$ gauged $SO(4)$ supergravity in $D = 4$ [9]; and maximal abelian truncations in $D = 4$, 5 and 7 [10]. One can also consider non-supersymmetric truncations of the maximal gauged supergravities. In [11, 12] the consistent truncations of the $D = 7$, $D = 5$ and $D = 4$ supergravities to the graviton plus scalar subsectors comprising only the diagonal scalars in the $SL(N, \mathbb{R})/SO(N)$ scalar submanifolds were considered (with $N = 5$, 6 and 8 respectively), and the consistent embeddings in $D = 11$ and $D = 10$ were constructed.

Although the full consistency of the S^7 and S^4 reductions of $D = 11$ supergravity has essentially been demonstrated, no similar complete result exists for the S^5 reduction of type IIB supergravity. The field content of the full $N = 8$ gauged supergravity [13] consists of gravity; fifteen $SO(6)$ gauge fields; twelve 2-form gauge potentials in the 6 and $\overline{6}$ representations of $SO(6)$; 42 scalars in the $1+1+20'+10+\overline{10}$ representations of $SO(6)$, and the fermionic superpartners. It is believed that this can arise from an S^5 reduction of type IIB supergravity; at the linearised level, the reduction Ansatz was given in [14]. However, at the full non-linear level, the only complete demonstrations so far are for the consistent embedding of the maximal abelian $U(1)^3$ truncation [10], the $N = 4$ gauged $SU(2) \times U(1)$ truncation [8], and the scalar truncation in [11, 12]. The full metric Ansatz was conjectured in [15].

In this paper, we obtain the consistent reduction Ansatz for a different truncation of the maximal gauged $D=5$ supergravity. One can consistently set the $1+1+10+\overline{10}$ of scalars to zero, at the same time as setting the $6+\overline{6}$ of 2-form potentials to zero. The bosons that remain, namely gravity, the 15 Yang-Mills fields, and the $20'$ of scalars, come just from the metric plus the self-dual 5-form sector of the original type IIB theory. We shall obtain complete results for the consistent embedding of this subsector of the gauged $N=8$ theory, with all fifteen of the $SO(6)$ gauge fields $A^{ij}_{(1)}$, and the twenty scalars that parameterise the full $SL(6,\mathbb{R})/SO(6)$ submanifold of the complete scalar coset. These can be parameterised by a unimodular symmetric tensor T_{ij}.

Another way of expressing the truncation that we shall consider in this paper is as follows. The type IIB theory itself can be consistently truncated in $D=10$ so that just gravity and the self-dual 5-form remain. The fifteen Yang-Mills gauge fields and twenty scalars that we retain in our Ansatz are the full set of massless fields associated with Kaluza-Klein reduction from this ten-dimensional starting point. (The counting of massless fields is the same as the one that arises from a toroidal reduction from the same ten-dimensional starting point.) We shall see below that the self-duality condition on the 5-form plays an essential rôle in the consistency of the S^5 reduction.

This subsector of type IIB supergravity is particularly relevant for the AdS/CFT correspondence [16, 17, 18], because it is the metric and the self-dual 5-form that couple to the D3-brane.

We also address the more general question of the circumstances under which a theory admits a consistent sphere reduction. We show that in a D-dimensional theory of gravity coupled to an n-form field strength, a consistent S^n reduction that retains the full set of $SO(n+1)$ Yang-Mills fields together with coupled massless scalars is possible only when $D=11$ and $n=4$ or 7, or in $D=10$ with $n=5$. Furthermore, in $D=11$ the theory has to be the bosonic sector of eleven-dimensional supergravity, with the FFA term, while in $D=10$ the 5-form must be self-dual. In all three cases the full set of massless scalars includes a subset T_{ij} described by the coset $SL(n+1,\mathbb{R})/SO(n+1)$. (Such a coset structure is absent for any other values of (D,n), and so it would not be appropriate to look for a consistent reduction Ansatz with scalars T_{ij} of $SL(n+1,\mathbb{R})/SO(n+1)$ for generic (D,n).) Of the three cases where such a consistent sphere reduction is possible, the ten-dimensional one is singled out as the only case where the consistent reduction includes only gravity plus the $SO(n+1)$ gauge fields $A^{ij}_{(1)}$ and the scalars T_{ij}. By contrast, for $D=11$ reduced on S^4 one must additionally retain a set of five 2-form potentials, while for $D=11$ reduced on S^7

one must instead additionally retain 35 pseudoscalars as well as the 35 scalars T_{ij}, in order to achieve consistent reductions.[1]

2 The $SO(6)$ reduction Ansatz on S^5

We parameterise the fields for this truncated theory as follows. The twenty scalars, which are in the $20'$ representation of $SO(6)$, are represented by the symmetric unimodular tensor T_{ij}, where i is a 6 of $SO(6)$. The fifteen $SO(6)$ Yang-Mills gauge fields will be represented by the 1-form potentials $A^{ij}_{(1)}$, antisymmetric in i and j. The inverse of the scalar matrix T_{ij} is denoted by T^{-1}_{ij}. In terms of these quantities, we find that the Kaluza-Klein reduction Ansatz is given by

$$\begin{aligned}
d\hat s_{10}^2 &= \Delta^{1/2}\, ds_5^2 + g^{-2}\, \Delta^{-1/2}\, T^{-1}_{ij}\, D\mu^i\, D\mu^j\,, &(1)\\
\hat H_{(5)} &= \hat G_{(5)} + \hat{*}\hat G_{(5)}\,, &(2)\\
\hat G_{(5)} &= -g\, U\, \epsilon_{(5)} + g^{-1}\, (T^{-1}_{ij}\, {*D\, T_{jk}}) \wedge (\mu^k\, D\mu^i)\\
&\quad -\tfrac12 g^{-2}\, T^{-1}_{ik}\, T^{-1}_{j\ell}\, {*F_{(2)}}^{ij} \wedge D\mu^k \wedge D\mu^\ell\,, &(3)\\
\hat{*}\hat G_{(5)} &= \frac{1}{5!}\, \epsilon_{i_1\cdots i_6}\, \Big[g^{-4}\, U\, \Delta^{-2}\, D\mu^{i_1} \wedge \cdots \wedge D\mu^{i_5}\, \mu^{i_6}\\
&\quad -5 g^{-4}\, \Delta^{-2}\, D\mu^{i_1} \wedge \cdots \wedge D\mu^{i_4} \wedge DT_{i_5 j}\, T_{i_6 k}\, \mu^j\, \mu^k\\
&\quad -10 g^{-3}\, \Delta^{-1}\, F^{i_1 i_2}_{(2)} \wedge D\mu^{i_3} \wedge D\mu^{i_4} \wedge D\mu^{i_5}\, T_{i_6 j}\, \mu^j \Big]\,, &(4)
\end{aligned}$$

where

$$\begin{aligned}
&U \equiv 2 T_{ij}\, T_{jk}\, \mu^i\, \mu^k - \Delta\, T_{ii}\,, \qquad \Delta \equiv T_{ij}\, \mu^i\, \mu^j\,,\\
&F^{ij}_{(2)} = dA^{ij}_{(1)} + g\, A^{ik}_{(1)} \wedge A^{kj}_{(1)}\,, \qquad DT_{ij} \equiv dT_{ij} + g\, A^{ik}_{(1)}\, T_{kj} + g\, A^{jk}_{(1)}\, T_{ik}\,,\\
&\mu^i\, \mu^i = 1\,, \qquad D\mu^i \equiv d\mu^i + g\, A^{ij}_{(1)}\, \mu^j\,, \qquad (5)
\end{aligned}$$

and $\epsilon_{(5)}$ is the volume form on the five-dimensional spacetime. Note that $\hat{*}\hat G_{(5)}$ is derivable from the given expressions (1) and (3); we have presented it here because it is quite an involved computation. The coordinates μ^i, subject to the constraint $\mu^i\,\mu^i = 1$, parameterise points in the internal 5-sphere. In obtaining the above Ansatz we have been guided by previous results in the literature, including the S^4 reduction Ansatz from $D = 11$ that was constructed in [4, 5].

[1]This is related to the fact that one can consistently truncate five-dimensional maximal gauged supergravity to the $SO(6)$ gauge fields plus scalars of the coset $SL(6,\mathbb{R})/SO(6)$. By contrast, the analogous truncations cannot be performed in the maximal gauged supergravities in $D = 7$ and $D = 4$.

It is consistent to truncate the fields of type IIB supergravity to the metric and self-dual 5-form $\hat H_{(5)}$. The ten-dimensional equations for motion for these fields are then given by

$$\begin{aligned} \hat R_{MN} &= \frac{1}{96}\,\hat H_{MPQRS}\,\hat H_N{}^{PQRS}\,,\\ d\hat H_{(5)} &= 0\,. \end{aligned} \tag{6}$$

The Ansatz presented above satisfies these equations of motion if and only if the five-dimensional fields satisfy the equations

$$\begin{aligned} D(T^{-1}_{ik}\,T^{-1}_{j\ell}\,{*F}^{k\ell}_{(2)}) &= -2g\,T^{-1}_{k[i}\,{*DT}_{j]k} - \tfrac18\epsilon_{ijk_1\cdots k_4}\,F^{k_1k_2}_{(2)}\wedge F^{k_3k_4}_{(2)}\,,\\ D(T^{-1}_{ik}\,{*DT}_{kj}) &= -2g^2\,(2T_{ik}\,T_{jk} - T_{ij}\,T_{kk})\,\epsilon_{(5)} + T^{-1}_{ik}\,T^{-1}_{\ell m}\,{*F}^{\ell k}_{(2)}\wedge F^{mj}_{(2)}\\ &\quad -\tfrac16\delta_{ij}\Big[-2g^2\,(2T_{k\ell}\,T_{k\ell} - (T_{kk})^2)\,\epsilon_{(5)} + T^{-1}_{pk}\,T^{-1}_{\ell m}\,{*F}^{\ell k}_{(2)}\wedge F^{mp}_{(2)}\Big]\,, \end{aligned} \tag{7}$$

together with the five-dimensional Einstein equation. These equations of motion can all be derived from the five-dimensional Lagrangian

$$\begin{aligned} \mathcal{L}_5 &= R\,{*1\!\!1} - \tfrac14 T^{-1}_{ij}\,{*DT}_{jk}\wedge T^{-1}_{k\ell}\,DT_{\ell i} - \tfrac14 T^{-1}_{ik}\,T^{-1}_{j\ell}\,{*F}^{ij}_{(2)}\wedge F^{k\ell}_{(2)} - V\,{*1\!\!1}\\ &\quad -\tfrac{1}{48}\,\epsilon_{i_1\cdots i_6}\Big(F^{i_1i_2}_{(2)}\,F^{i_3i_4}_{(2)}\,A^{i_5i_6}_{(1)} - g\,F^{i_1i_2}_{(2)}\,A^{i_3i_4}_{(1)}\,A^{i_5j}_{(1)}\,A^{ji_6}_{(1)} + \tfrac25 g^2\,A^{i_1i_2}_{(1)}\,A^{i_3j}_{(1)}\,A^{ji_4}_{(1)}\,A^{i_5k}_{(1)}\,A^{ki_6}_{(1)}\Big)\,, \end{aligned} \tag{8}$$

where the potential V is given by

$$V = \tfrac12 g^2\Big(2T_{ij}\,T_{ij} - (T_{ii})^2\Big)\,. \tag{9}$$

In (8) we have omitted the wedge symbols in the final topological term, to economise on space. The Lagrangian is in agreement with the one for five-dimensional gauged $SO(6)$ supergravity in [13].

To see this, we look first at the ten-dimensional equation $d\hat H_{(5)} = 0$, with the Ansatz given above. The terms involving structures of the form $X^{ij}_{(4)}\wedge D\mu^i\wedge D\mu^j$, where $X^{ij}_{(4)}$ represents a 4-form in the five-dimensional spacetime, give rise to the Yang-Mills equations above. Contributions of this kind come from $d\hat G_{(5)}$ and also from the final term in $d\hat{*}\hat G_{(5)}$. The terms involving structures of the form $X^{ij}_{(5)}\wedge(\mu^i\,D\mu^j)$, where $X^{ij}_{(5)}$ represents a 5-form in the five-dimensional spacetime, give rise to the scalar equations of motion in (7). Since $\mu^i\,D\mu^i = \mu^i\,d\mu^i = \frac12 d(\mu^i\,\mu^i) = 0$, there is a trace subtraction in the scalar equation, and we read off $X^{ij}_{(5)} - \frac16\delta_{ij}X^{kk}_{(5)} = 0$ as the five-dimensional equation of motion. Contributions of this structure come only from $d\hat G_{(5)}$. Finally, all other structures arising from calculating $d\hat H_5 = 0$ vanish identically, without the use of any five-dimensional equations of motion. In deriving these results one needs to make extensive use of the Schoutens over-antisymmetrisation identity, $\epsilon_{[i_1\cdots i_6}\,V_{i_7]} = 0$.

It is worth remarking that it is essential for the consistency of the reduction Ansatz that the 5-form $\hat{H}_{(5)}$ should be self-dual. One cannot simply consider a reduction Ansatz for a ten-dimensional theory consisting of gravity plus a non-self-dual 5-form, whose Ansatz is given by $\hat{G}_{(5)}$ in (3). Although the Bianchi identity $d\hat{G}_{(5)} = 0$ would give perfectly acceptable equations of motion for $F_{(2)}^{ij}$ and T_{ij}, the field equation $d\hat{*}\hat{G}_{(5)} = 0$ would produce the (unacceptable) constraint

$$\epsilon_{ijk_1\cdots k_4}\, F_{(2)}^{k_1k_2} \wedge F_{(2)}^{k_3k_4} = 0\,. \tag{10}$$

It is only by combining $\hat{G}_{(5)}$ and $\hat{*}\hat{G}_{(5)}$ together into the self-dual field $\hat{H}_{(5)}$ that a consistent five-dimensional result is obtained, with (10) now combining with terms from $d\hat{G}_{(5)}$ to form part of the five-dimensional Yang-Mills equations given in (7).[2] It is interesting, therefore, that self-duality of the 5-form is apparently forced on us by the requirements of Kaluza-Klein consistency, if we try to "invent" a ten-dimensional theory that can be reduced on S^5. Thus once again we see that supersymmetry and Kaluza-Klein consistency for sphere reductions seem to go hand in hand.[3]

The Kaluza-Klein S^5 reduction that we have obtained here retains the full set of massless fields that can result from the reduction of gravity plus a self-dual 5-form in $D = 10$. In other words, after the initial truncation of the type IIB theory in $D = 10$, no further truncation of massless fields has been performed.

It should also be emphasised that it would be inconsistent to omit the fifteen $SO(6)$ gauge fields when considering the embedding of the twenty scalars T_{ij}. This can be seen from the Yang-Mills equations in (7), which have a source term $g\, T^{-1}_{k[i}\, {*DT_{j]k}}$ appearing on the right-hand side. This is a quite different situation from a toroidal reduction, where it is always consistent to truncate to the scalar sector, setting the gauge fields to zero. The new feature here in the sphere reduction is that the scalar fields are charged under the gauge group. This is a general feature of all the sphere reductions, to $D = 4$, $D = 5$ and $D = 7$, and thus in all cases it is inconsistent to include the full set of scalar fields without including

[2] If one were to consider the reduction of a ten-dimensional theory with a non-self-dual 5-form there would be additional fields present in a complete massless truncation. These would comprise $10 = 4 + 6$ vector potentials and $5 = 1 + 4$ scalars. However the inclusion of these fields would still not achieve a consistent reduction Ansatz, since the current (10), which is in the 15 of $SO(6)$, could still not acquire an interpretation as a source term for the additional fields. Thus the additional requirement of self-duality seems to be essential for consistency. (Of course anti-self-duality would be equally good.)

[3] Of course the initial truncation of the type IIB theory to its gravity plus self-dual 5-form sector is itself a non-supersymmetric one, but the crucial point is that the consistency of the Kaluza-Klein S^5 reduction is singling out a starting point that is itself a subsector of a *supersymmetric* theory.

the gauge fields as well (see also [4, 5, 19]). (One *can* consistently truncate to the diagonal scalars in T_{ij}, setting all the gauge fields to zero, as in [11, 12], since then $T^{-1}_{k[i}\,{*DT_{j]k}} = 0$.)

Our testing of the consistency of the reduction Ansatz (1)-(4) has so far been restricted to checking the equations of motion for $\hat{H}_{(5)}$. A full testing of the consistency of the ten-dimensional Einstein equations would be quite involved, and will be addressed in future work.[4] In the next section we shall show that the reduction Ansatz that we have obtained in this paper reduces, with appropriate additional truncations, to results that have been obtained previously. Since the complete consistency was proven in these earlier results, including the ten-dimensional Einstein equations, this provides further supporting evidence for the complete consistency of the Ansatz that we have constructed here.

Finally in this section, we remark that the S^5 reduction that we have constructed here is also consistent if we include the dilaton $\hat{\phi}$ and axion $\hat{\chi}$ of the type IIB theory. These simply reduce according to the Ansätze

$$\hat{\phi} = \phi\,, \qquad \hat{\chi} = \chi\,, \tag{11}$$

where the unhatted quantities denote the fields in five dimensions. In this reduction they do not appear in the previous Ansätze for the metric and self-dual 5-form, and in $D = 5$ they just give rise to the additional $SL(2,\mathbb{R})$-invariant Lagrangian

$$\mathcal{L}_{(\phi,\chi)} = -\tfrac{1}{2}{*d\phi}\wedge d\phi - \tfrac{1}{2}e^{2\phi}\,{*d\chi}\wedge d\chi\,, \tag{12}$$

which is added to (8). Note in particular that ϕ and χ do not appear in the five-dimensional scalar potential V.

3 Truncations to previous results

We can consider three different truncations of the reduction scheme of the previous section, in order to make contact with previous results in the literature. The first of the three involves truncating the twenty scalars T_{ij} of the $SL(6,\mathbb{R})/SO(6)$ coset to the diagonal subset

$$T_{ij} = \mathrm{diag}\,(X_1, X_2, X_3, X_4, X_5, X_6)\,, \tag{13}$$

[4] The experience in all previous work on consistent reductions is that although the actual checking of the higher-dimensional Einstein equations is the most difficult part from a computational point of view, the consistency seems to be assured once it has been achieved for the equations of motion for the antisymmetric tensor fields, which itself is an extremely stringent requirement.

where $\prod_i X_i = 1$, and setting all the fifteen gauge fields $F^{ij}_{(2)}$ to zero. This reduces to the embedding that was obtained in [11], for which the complete proof of consistency was constructed in [12].

The second possible truncation involves reducing the scalar sector still further, to

$$T_{ij} = \mathrm{diag}\,(\widetilde X_1, \widetilde X_1, \widetilde X_2, \widetilde X_2, \widetilde X_3, \widetilde X_3)\,, \tag{14}$$

where $\prod_a \widetilde X_a = 1$, but now retaining the three $U(1)$ gauge fields $F^{12}_{(2)}$, $F^{34}_{(2)}$ and $F^{56}_{(2)}$ of the maximal abelian $U(1)^3$ subgroup of $SO(6)$. (It is easy to see, by looking at the five-dimensional equations of motion in (7), that this is a consistent truncation.) The truncated theory is supersymmetric, and describes five-dimensional $U(1)$-gauged simple supergravity coupled to two $U(1)$ vector multiplets. The consistent embedding of this theory in type IIB supergravity was obtained in [10].

The third possible truncation involves retaining just a single scalar field X, by taking

$$T_{ij} = \mathrm{diag}\,(X, X, X, X, X^{-2}, X^{-2})\,, \tag{15}$$

at the same time retaining only the gauge fields of $SU(2)\times U(1)$. It is convenient now to take the $SO(6)$ indices i to range over $0 \le i \le 5$. We take all the gauge potentials $A^{ij}_{(1)}$ to be zero except for the following:

$$\begin{aligned}
&A^{01}_{(1)} = A^{23}_{(1)} = \tfrac{1}{\sqrt2}\,A^1_{(1)}\,,\quad A^{02}_{(1)} = A^{31}_{(1)} = -\tfrac{1}{\sqrt2}\,A^2_{(1)}\,,\quad A^{03}_{(1)} = A^{12}_{(1)} = \tfrac{1}{\sqrt2}\,A^3_{(1)}\,,\\
&A^{45}_{(1)} = B_{(1)}\,.
\end{aligned} \tag{16}$$

We also parameterise the coordinates μ^i of the internal 5-sphere as follows:

$$\begin{aligned}
&\mu^0 + \mathrm{i}\,\mu^3 = \cos\xi\,\cos\tfrac12\theta\,e^{\mathrm{i}\,(\psi+\phi)/2}\,,\qquad \mu^1 + \mathrm{i}\,\mu^2 = \cos\xi\,\sin\tfrac12\theta\,e^{\mathrm{i}\,(\psi-\phi)/2}\,,\\
&\mu^4 + \mathrm{i}\,\mu^5 = \sin\xi\,e^{\mathrm{i}\,\tau}\,.
\end{aligned} \tag{17}$$

Substituting (15), (16) and (17) into the metric Ansatz (1), we obtain

$$\begin{aligned}
d\hat s_{10}^2 &= \Delta^{1/2}\,ds_5^2 + g^{-2}\,X\,\Delta^{1/2}\,d\xi^2 + g^{-2}\Delta^{-1/2}\,X^2\,s^2\,\Big(d\tau - g\,B_{(1)}\Big)^2\\
&\quad +\tfrac14 g^{-2}\,\Delta^{-1/2}\,X^{-1}\,c^2\,\sum_i (\sigma^i - \sqrt2 g\,A^i_{(1)})^2\,,
\end{aligned} \tag{18}$$

where $c\equiv\cos\xi$, $s\equiv\sin\xi$, $\Delta = X^{-2}\,s^2 + X\,c^2$, $h^i \equiv \sigma^i - \sqrt2 g\,A^i_{(1)}$, and the σ_i denote the three left-invariant 1-forms of $SU(2)$, given by $\sigma_1 + \mathrm{i}\,\sigma_2 = e^{-\mathrm{i}\psi}\,(d\theta + \mathrm{i}\,\sin\theta\,d\phi)$, $\sigma_3 = d\psi + \cos\theta\,d\phi$. This is the metric Ansatz obtained in [8] for the embedding of five-dimensional $N=4$ gauged $SU(2)\times U(1)$ supergravity in $D=10$. The internal 5-sphere now has a geometrical interpretation as a foliation by $S^3\times S^1$, with ξ parameterising the foliating surfaces.

Substituting (15), (16) and (17) into the Ansatz for $\hat{G}_{(5)}$ given in (3), leads to

$$\begin{aligned} \hat{G}_{(5)} &= -g\,U\,\varepsilon_5 - \frac{3sc}{g} X^{-1} {*dX} \wedge d\xi + \frac{c^2}{8\sqrt{2}\,g^2} X^{-2} {*F^i_{(2)}} \wedge h^j \wedge h^k\,\varepsilon_{ijk} \\ &\quad -\frac{s\,c}{2\sqrt{2}\,g^2} X^{-2} {*F^i_{(2)}} \wedge h^i \wedge d\xi - \frac{sc}{g^2} X^4 {*G_{(2)}} \wedge d\xi \wedge (d\tau - gB_{(1)})\,, \end{aligned} \tag{19}$$

where $U = -2(X^2\,c^2 + X^{-1}\,s^2 + X^{-1})$, again in agreement with the Ansatz obtained in [8]. As a consistency check, we can also verify that substituting (15), (16) and (17) into the Ansatz (4) for $\hat{*}\hat{G}_{(5)}$ gives the same expression as the one obtained for $\hat{*}\hat{G}_{(5)}$ in [8]. Thus we have verified that the gauged $SO(6)$ embedding that we have obtained in this paper can be truncated to the $SU(2)\times U(1)$ embedding of the $N=4$ theory whose consistency was proven in [8]. This again provides further supporting evidence for the consistency of the our gauged $SO(6)$ reduction Ansatz.

4 Consistency conditions for sphere reductions

One of the interesting outcomes from our analysis is that it is essential for the consistency of the 5-form reduction Ansatz that it be a *self-dual* 5-form, rather than an unconstrained one. It seems, therefore, that the requirement of consistency of the sphere reduction has singled out a ten-dimensional starting point that is itself embeddable in a supersymmetric theory.

This raises the more general question of what possible higher-dimensional theories might allow consistent Kaluza-Klein sphere reductions. All the known examples are associated with supersymmetric higher-dimensional theories, but one might wonder whether this was just a reflection of the fact that these are the cases that have received the most attention in the literature. However, the following argument seems to suggest that the supersymmetric cases may be the only ones that can allow consistent S^n sphere reductions, in which all the massless fields (including the $SO(n+1)$ Yang-Mills fields) are retained.

Consider a D-dimensional theory of gravity plus an n-form field strength, with the Lagrangian

$$e^{-1}\,\mathcal{L}_D = R - \tfrac{1}{2\,n!}\,F_n^2\,, \tag{20}$$

where $e=\sqrt{-g}$. If this were to give a $(D-n)$-dimensional theory with an $SO(n+1)$ gauge group, as a consistent reduction on S^n, it would be necessary that the ungauged $(D-n)$-dimensional theory obtained by reducing instead on the torus T^n should have a global symmetry group G that contains $SO(n+1)$ as a compact subgroup, since an $SO(n+1)$ factor in the denominator group would be gauged in the spherical reduction. A

reduction on T^n always produces a theory with a $GL(n, \mathrm{I\!R})$ global symmetry, which has $SO(n)$ as its maximal compact subgroup, and so this would be insufficient for allowing an $SO(n+1)$ gauging. In special cases the $GL(n, \mathrm{I\!R})$ global symmetry can be enhanced [20] to $SL(n+1, \mathrm{I\!R})$, but this happens only if there is a "conspiracy" between axionic scalars coming from the metric and axions coming from the form-field F_n. For this conspiracy to occur, the strengths of the dilaton couplings to axions from these two sources must be the same.[5] Specifically, if $\vec{\phi}$ denotes the set of n canonically-normalised dilatons that result from the T^n reduction then all the dilaton/axion couplings should be of the form $e^{\vec{a}_i \cdot \vec{\phi}} (\partial \chi_i)^2$, where the constant vectors $\vec{a}_i$ satisfy $(\vec{a}_i)^2 = 4$ for each i. (In fact the full set of $\vec{a}_i$ vectors would constitute the positive-root vectors of $SL(n+1, \mathrm{I\!R})$ [20].) It was shown in [21] that the strengths of dilaton couplings can be conveniently characterised in terms of the quantity Δ, related to the dilaton vector $\vec{a}$ by

$$\vec{a}^2 = \Delta - \frac{2(n-1)(D-n-1)}{D-2}\,, \tag{21}$$

for an n-form field strength in D dimensions, since the quantity Δ is preserved under toroidal Kaluza-Klein reduction. Furthermore, it was shown that the Kaluza-Klein vectors and axions coming from a toroidal reduction of the metric always have $\Delta = 4$. It therefore follows that for the symmetry enhancement to $SL(n+1, \mathrm{I\!R})$ to take place, the dilaton-coupling Δ for the original n-form field[6] in (20) must also take the value $\Delta = 4$. An enumeration of all possible cases shows that for Lagrangians of the form (20) one gets $\Delta = 4$ only for

$$(D, n) = (11, 4)\,, \qquad (11, 7)\,, \qquad (10, 5)\,. \tag{22}$$

The above considerations seem to single out the three cases in (22) as the only ones where an S^n reduction of a Lagrangian of the form (20) could consistently yield the gauge fields of $SO(n+1)$ and the scalars T_{ij} of the coset $SL(n+1, \mathrm{I\!R})/SO(n+1)$.[7] As we have seen in section 2 of this paper, it can turn out that additional structure is also required in order to achieve consistency, namely the self-duality of the 5-form in the case of $(D, n) = (10, 5)$.

[5] It is always the case that the counting of dilatons and axions would be consistent with the numerology required for an $SL(n+1, \mathrm{I\!R})/SO(n+1)$ coset structure, but this does not in general imply that the enhanced coset actually occurs.

[6] If there is no dilaton in the original higher-dimensional theory then the "dilaton coupling" Δ is given by setting $\vec{a}^2 = 0$ in (21), and it is this value that is preserved under toroidal reduction.

[7] Consistent sphere reductions with scalars T_{ij} for more general (D, n) values for the Lagrangian (20) have been suggested in [19]. We expect that a more complete analysis of the consistency of the reduction would exclude such possibilities. In particular, scalars T_{ij} parametrising the coset $SL(n+1, \mathrm{I\!R})/SO(n+1)$ only occur in the three cases listed in (22).

By the same token one can expect that a consistent reduction would only be possible in the cases $(D, n) = (11, 4)$ and $(11, 7)$ if additional structure is also present in the eleven-dimensional Lagrangian.

For example, if we consider the S^7 reduction from $D = 11$, then the total set of 70 spin-0 fields decompose as a 35_v of scalars and a 35_c of pseudoscalars. The 35 scalars in T_{ij} correspond to the 35_v. Turning on these forces all the 28 gauge fields of $SO(8)$ to be excited, and in turn these excite the remaining 35_c of pseudoscalars. In order for an S^7 reduction that retains all these fields to be consistent, it is necessary to include an $F_{(4)} \wedge F_{(4)} \wedge A_{(3)}$ term in the original Lagrangian (20), with precisely the coefficient dictated by $D = 11$ supersymmetry. This point also emphasises that in the S^7 reduction one cannot consider just the 35 scalars T_{ij} in isolation; the full set of bosonic fields of $N = 8$ gauged supergravity (and not merely the 28 gauge fields) must be included if the full set of T_{ij} scalars are present.

A similar situation arises with the S^4 reduction from $D = 11$. Including the full set of 14 scalars T_{ij} in a consistent reduction will force the complete set of massless fields to be non-vanishing, including not only the ten Yang-Mills fields of $SO(5)$ but also the five 3-form field strengths coming from the antisymmetric tensor. The consistency of the reduction is then only possible if the FFA term of $D = 11$ supergravity is included.

Another possibility for obtaining further examples of consistent sphere reductions is to include a dilaton in the higher-dimensional Lagrangian, whose coupling to the n-form field strength is arranged to have $\Delta = 4$:

$$e^{-1}\,\mathcal{L}_D = R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2\,n!}\,e^{a\,\phi}\,F_n^2\,, \tag{23}$$

with $a^2 = 4 - 2(n-1)(D-n-1)/(D-2)$. (Lagrangians of this type without the restriction on the value of the constant a have also been discussed in [19] in the context of sphere reductions.) In this case there will no longer be an $\mathrm{AdS}_{D-n} \times S^n$ vacuum solution, but rather a warped product of a domain wall and S^n. The possibilities for achieving $\Delta = 4$ couplings are in fact rather limited, given that a should be a real number. First of all, for an n-form with $4 \le n \le D-4$ it can be seen that we must have $D \le 11$. For example if $n = 4$, then we must have $D \le 11$, while for $n = 5$ we must have $D \le 10$. (It is only necessary to consider forms with $n \le D/2$ since Hodge duality maps those with $n > D/2$ into this range.) Interestingly, for $n = 3$ one can achieve $\Delta = 4$ in an arbitrary dimension D; the Lagrangian (23) then corresponds to the low-energy effective theory of the D-dimensional bosonic string. We can expect that the full 3-sphere reduction of this Lagrangian (keeping the complete $SO(4)$ gauge fields of its isometry group, not merely the $SU(2)$ subset of left-invariant fields) will be consistent. Similarly, we can expect that it should be consistent to

reduce the theory with $n = 3$ on a $(D-3)$-sphere. Before the gauging, the scalar coset in $D = 3$ is $SO(D-2, D-2)/(SO(D-2) \times SO(D-2))$. One of the $SO(D-2)$ denominator group factors can be gauged, and we obtain the scalar coset $SL(D-2, \mathbb{R})/SO(D-2)$ together with the additional gauge fields of $SO(D-2)$, and a singlet scalar (which is the original dilaton of the D-dimensional theory).

A consistent sphere reduction, albeit of a slightly different kind, has in fact been obtained in an example where there is a dilaton in the higher-dimensional theory that couples to the form-field. In [6] it was shown that a reduction of the massive type IIA supergravity on an internal 4-dimensional space that is locally S^4 gives rise to the $N = 2$ $SU(2)$ gauged supergravity in $D = 6$.

5 Conclusions and further comments

In this paper we have constructed a consistent Kaluza-Klein reduction Ansatz for embedding the subset of the fields of five-dimensional $N = 8$ gauged supergravity, comprising gravity, the fifteen $SO(6)$ gauge fields, and the twenty scalars of the $SL(6, \mathbb{R})/SO(6)$ submanifold of the full scalar manifold, into type IIB supergravity in $D = 10$. This embedding can equivalently be viewed as a complete reduction Ansatz (with no truncation of massless fields) for the ten-dimensional theory comprising just gravity plus a self-dual 5-form, which itself is a consistent truncation of type IIB supergravity.

A crucial point in the analysis is that in the gauged five-dimensional supergravity one cannot consistently set the Yang-Mills fields to zero, while retaining the full set of scalar fields, unlike the situation in ungauged supergravity. In the context of the truncation that we consider in this paper, where we retain the twenty scalars T_{ij}, we cannot ignore the fifteen $SO(6)$ gauge fields, since the scalars act as sources for them.[8] We saw that the self-duality of the 5-form field of the type IIB theory plays an essential rôle in the consistency of the reduction.

More generally, we showed that if one starts from a theory comprising gravity and an n-form field strength, then a consistent reduction on S^n that retains the scalars T_{ij} of the coset $SL(n+1, \mathbb{R})/SO(n+1)$ will also have to include at least the gauge fields of $SO(n+1)$, and furthermore will only be possible for the S^4 and S^7 reductions of $D = 11$, and the S^5 reduction of $D = 10$. In fact the consistency will in addition require that the theories

[8]This also implies that solutions built using these scalars will in general require non-vanishing Yang-Mills fields.

in $D = 11$ and $D = 10$ have the additional structures associated with $D = 11$ and type IIB supergravity, namely the FFA term in $D = 11$, and the self-duality of the 5-form in $D = 10$. In $D = 7$ the scalars and Yang-Mills fields must be supplemented by the five 2-form potentials, while in $D = 4$ they must be supplemented by the 35 pseudo-scalars, in order to achieve consistency. On the other hand in $D = 5$ no additional fields beyond the scalars T_{ij} and the gauge fields $A_{(1)}^{ij}$ are required for consistency. In fact in all three cases this is related to the presence of the terms of the form (10), bilinear in Yang-Mills fields. In $D = 7$ this term acts as a source for the five 3-form fields; in $D = 4$ it acts as a source for the 35 pseudo-scalars; but in $D = 5$ it acts as a "source" for the Yang-Mills fields themselves. This special feature of the $D = 5$ gauged supergravity may have implications in the dual four-dimensional $N = 4$ super Yang-Mills theory.

We also discussed the more general possibilities that might arise if one includes a dilaton in the higher-dimensional theory. The possibilities for further examples of consistent sphere reductions seem to be rather limited, as discussed in section 4.

In a full S^5 reduction of type IIB supergravity there will be additional fields coming from the reduction of the NS-NS and R-R 2-form potentials, and from the dilaton and axion. A complete analysis of the S^5 reduction can therefore be expected to be extremely complicated. In particular, for example, the 10 and $\overline{10}$ of pseudo-scalars lead to a considerably more complicated metric reduction Ansatz. The Ansatz for a subset of the fields that included one scalar and one pseudoscalar was derived in [22], and in [23]. However even in that case, the construction of the Ansatz for the antisymmetric tensor fields is rather involved, and has not yet been pushed to completion.

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